

# LIFESPAN OF STRONG SOLUTIONS TO THE PERIODIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. An explicit lifespan estimate is presented for the derivative Schrödinger equations with periodic boundary condition.

## 1. INTRODUCTION

We consider the Cauchy problem for the following derivative nonlinear Schrödinger (DNLS) equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda \partial_x (|u|^{p-1} u), & t \in [0, T), \quad x \in \mathbb{T}, \\ u(0) = u_0, & x \in \mathbb{T} \end{cases} \quad (1)$$

on one-dimensional torus  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ , where  $p > 1$  and  $\lambda \in \mathbb{C} \setminus \{0\}$ . The aim of this paper is to study an explicit upper bound of lifespan of solutions for (1) in terms of the data  $u_0$  in the case  $\operatorname{Re} \lambda \neq 0$ .

The original DNLS equation on  $\mathbb{R}$  with  $p = 3$  and  $\lambda = -i$  with additional terms was derived in plasma physics for a model of Alfvén wave (see [13, 17]). By a simple computation, if  $\lambda \in i\mathbb{R}$ , then we have the charge ( $L^2$ ) conservation law for solutions of (1) and

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda \partial_x (|u|^{p-1} u), & t \in [0, T), \quad x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R} \end{cases} \quad (2)$$

with any  $p > 1$ . For the solution  $u$  for (2) with  $p = 3$  and  $\lambda = -i$ , the gauge transformed solution  $v$  defined by

$$v(t, x) = u(t, x) \exp \left( \frac{i}{2} \int_{-\infty}^x |u(t, y)|^2 dy \right)$$

satisfies

$$i\partial_t v + \partial_x^2 v = -i|v|^2 \partial_x v, \quad t \in [0, T), \quad x \in \mathbb{R}. \quad (3)$$

Similarly, in the case of (1) with  $p = 3$  and  $\lambda = -i$ , the gauge transformed solution  $w$  defined by

$$w(t, x) = u(t, x) \exp \left( \frac{i}{2} \int_0^x |u(t, y)|^2 dy - \frac{i}{2} \int_0^t \operatorname{Im} \left( \overline{u(t', 0)} \partial_x u(t', 0) \right) + 4|u(t', 0)|^4 dt' \right)$$

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satisfies

$$i\partial_t w + \partial_x^2 w = -i|w|^2 \partial_x w, \quad t \in [0, T), \quad x \in \mathbb{T}. \quad (4)$$

Then the following energies are conserved:

$$\begin{aligned} E_1(v(t)) &= \int_{\mathbb{R}} \left( |\partial_x v(t)|^2 + \frac{1}{2} \operatorname{Im} |v(t)|^2 \overline{v(t)} \partial_x v(t) \right) dx = E_1(v(0)), \\ E_2(w(t)) &= \int_{\mathbb{T}} \left( |\partial_x w(t)|^2 + \frac{1}{2} \operatorname{Im} |w(t)|^2 \overline{w(t)} \partial_x w(t) \right) dx = E_2(w(0)). \end{aligned}$$

The well-posedness of (2) with  $p = 3$  and  $\lambda = -i$  has been studied by many authors. For example, Tsutsumi and Fukuda showed local well-posedness in the Sobolev space  $H^s(\mathbb{R}) = (1 - \Delta)^{-s/2} L^2(\mathbb{R})$  with  $s > 3/2$  in [22]. Moreover, by using the gauge transformation, the  $H^s(\mathbb{R})$  local and global well-posedness with  $s \geq 1/2$  has been studied in [3, 8, 9, 10, 18]. Furthermore, Biagioni and Linares showed the  $H^s(\mathbb{R})$  ill-posedness with  $s < 1/2$  in [2]. This means  $H^{1/2}(\mathbb{R})$  gives the sharp criteria for the local well-posedness for (3). We also refer the reader to [7, 12, 16] for generalized results.

On the other hand, the well-posedness of the Cauchy problem (1) with  $p = 3$  and  $\lambda \in i\mathbb{R}$  has also been studied. Tsutsumi and Fukuda showed local well-posedness in the Sobolev space  $H^s(\mathbb{T}) = (1 - \Delta)^{-s/2} L^2(\mathbb{T})$  with  $s > 3/2$  in [22] as well as for (2). In [11], Herr showed the local and global well-posedness in  $H^s(\mathbb{T})$  with  $s \geq 1/2$  by using the modified gauge transformation. We also refer the reader to [1, 6, 14, 19, 21, 23] for generalized results.

Even though, local and global well-posedness for DNLS equation has been studied, the blowup of solution for DNLS is still open in a general setting, where the conservation law is insufficient or fails. Partial results have been obtained in [20]. In this article, we study the finite time blowup of solutions for (1) by using a simple ODE argument. See also [4, 5, 15].

An obvious global solution for (1) is  $u(t, x) = C$  for  $C \in \mathbb{C}$ . So it is necessary to consider a set of initial data without constants in order to show the finite time blowup of (1). Here we consider the initial data and solutions with vanishing total density defined as follows:

**Definition 1.** For  $u_0 \in H^2(\mathbb{T})$  satisfying  $\int_{\mathbb{T}} u_0(x) dx = 0$ ,  $u$  is called a strong solution with vanishing total density of the Cauchy problem (1) if there exists  $T \in (0, \infty]$  such that  $u \in C^1([0, T]; H^2(\mathbb{T}))$  satisfies (1) and  $\int_{\mathbb{T}} u(t, x) dx = 0$  for any  $t \in [0, T)$ .

**Remark 1.** Formally,

$$\frac{d}{dt} \int_{\mathbb{T}} u(t, x) dx = (2\pi)^{1/2} \frac{d}{dt} \hat{u}(t, 0) = -i(2\pi)^{1/2} \mathfrak{F}[-\partial_x^2 u + \lambda \partial_x(|u|^{p-1}u)](0) = 0.$$

This implies that if  $\int_{\mathbb{T}} u_0(x) dx = 0$ , then  $\int_{\mathbb{T}} u(t, x) dx = 0$  for any  $t \in [0, T)$ .

In this article, for  $H^2(\mathbb{T})$  initial data with vanishing total density, we assume the existence of strong solutions with vanishing total density. We define the lifespan  $T_0$  of a strong solution  $u$  to the Cauchy problem (1) by

$$T_0 = \sup\{T > 0; u \text{ is a strong solution for (1)}\}.$$

Then, from the ordinary differential inequality for  $\int_0^{2\pi} \int_0^x u(\cdot, x) \overline{u(\cdot, y)} dy dx$ , we may obtain the following equivalent conditions for the finite time blowup for (1) and estimate of lifespan.

**Proposition 2.** *Let  $u_0 \in L^2(\mathbb{T})$  satisfy  $\int_{\mathbb{T}} u_0(x) dx = 0$ . Then the following statements are equivalent:*

(i)  $u_0$  satisfies

$$\operatorname{Re} \lambda \cdot \operatorname{Im} \int_0^{2\pi} u_0(x) \int_0^x \overline{u_0(y)} dy dx > 0. \quad (5)$$

(ii) There exists  $\alpha \in \mathbb{C}$  such that

$$\operatorname{Re} \alpha \cdot \operatorname{Re} \lambda, \quad \operatorname{Im} \left( \alpha \int_0^{2\pi} u_0(x) \int_0^x \overline{u_0(y)} dy dx \right) > 0. \quad (6)$$

If  $u_0$  satisfies one of the equivalent conditions above and  $u_0 \in H^2(\mathbb{T})$ , then the corresponding strong solution with vanishing total density of the Cauchy problem (1) blows up in finite time. Moreover, the associated lifespan is estimated by

$$T_0 \leq \frac{(2\pi)^p}{(p-1)|\operatorname{Re} \lambda|} \left| \int_0^{2\pi} u_0(x) \int_0^x \overline{u_0(y)} dy dx \right|^{-\frac{p-1}{2}}.$$

**Remark 2.** For  $f \in L^2(\mathbb{T})$  with vanishing total density,

$$\begin{aligned} \operatorname{Re} \int_0^{2\pi} f(x) \int_0^x \overline{f(y)} dy dx &= \frac{1}{2} \int_0^{2\pi} \frac{d}{dx} \left| \int_0^x f(y) dy \right|^2 dx \\ &= \frac{1}{2} \left| \int_0^{2\pi} f(x) dx \right|^2 = 0. \end{aligned}$$

This means

$$\int_0^{2\pi} f(x) \int_0^x \overline{f(y)} dy dx \in i\mathbb{R}$$

and implies the equivalence between (5) and (6).

## 2. PROOF OF PROPOSITION 2

Let  $M(t) = \operatorname{Im} \left( \alpha \int_0^{2\pi} u(t, x) \int_0^x \overline{u(t, y)} dy dx \right)$ , where  $\alpha$  satisfies (6). Then  $M(t) > 0$  for sufficiently small  $t$ . By a direct calculation, we have

$$\begin{aligned} \frac{d}{dt} M(t) &= \operatorname{Im} \left( \alpha \int_0^{2\pi} \partial_t u(t, x) \int_0^x \overline{u(t, y)} dy dx \right) \\ &\quad + \operatorname{Im} \left( \alpha \int_0^{2\pi} u(t, x) \int_0^x \overline{\partial_t u(t, y)} dy dx \right) \\ &= I_1 + I_2. \end{aligned}$$

By the vanishing total density,  $I_1$  and  $I_2$  may be computed as follows:

$$\begin{aligned}
I_1 &= -\operatorname{Re} \left( \alpha \int_0^{2\pi} i \partial_t u(t, x) \int_0^x \overline{u(t, y)} dy dx \right) \\
&= -\operatorname{Re} \left( \alpha \int_0^{2\pi} \partial_x (-\partial_x u(t, x) + \lambda(|u(t, x)|^{p-1} u(t, x))) \int_0^x \overline{u(t, y)} dy dx \right) \\
&= -\operatorname{Re} \left( \alpha (-\partial_x u(t, 2\pi) + \lambda(|u(t, 2\pi)|^{p-1} u(t, 2\pi))) \int_0^{2\pi} \overline{u(t, y)} dy \right) \\
&\quad + \operatorname{Re} \left( \alpha \int_0^{2\pi} -\overline{u(t, x)} \partial_x u(t, x) + \lambda |u(t, x)|^{p+1} dx \right) \\
&= \operatorname{Re} \left( \alpha \int_0^{2\pi} -\overline{u(t, x)} \partial_x u(t, x) + \lambda |u(t, x)|^{p+1} dx \right), \\
I_2 &= \operatorname{Re} \left( \alpha \int_0^{2\pi} u(t, x) \int_0^x i \partial_t \overline{u(t, y)} dy dx \right) \\
&= \operatorname{Re} \left( \alpha \int_0^{2\pi} u(t, x) \overline{(-\partial_x u(t, x) + \lambda(|u(t, x)|^{p-1} u(t, x)))} dx \right) \\
&\quad - \operatorname{Re} \left( \alpha \overline{(-\partial_x u(t, 0) + \lambda(|u(t, 0)|^{p-1} u(t, 0)))} \int_0^{2\pi} u(t, x) dx \right) \\
&= \operatorname{Re} \left( \alpha \int_0^{2\pi} u(t, x) \overline{(-\partial_x u(t, x) + \lambda(|u(t, x)|^{p-1} u(t, x)))} dx \right).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{dt} M(t) &= -\operatorname{Re} \alpha \cdot \int_0^{2\pi} 2 \operatorname{Re}(u(t, x) \overline{\partial_x u(t, x)}) dx + 2 \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda \|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1} \\
&= -\operatorname{Re} \alpha \cdot \int_0^{2\pi} \partial_x |u(t, x)|^2 dx + 2 \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda \|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1} \\
&= 2 \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda \|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1}.
\end{aligned}$$

Since

$$|M(t)| \leq |\alpha| \|u(t)\|_{L^1(\mathbb{T})}^2 \leq (2\pi)^{\frac{2p}{p+1}} |\alpha| \|u(t)\|_{L^{p+1}(\mathbb{T})}^2,$$

we have

$$\frac{d}{dt} M(t) \geq 2(2\pi)^{-p} |\alpha|^{-\frac{p+1}{2}} \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda M(t)^{\frac{p+1}{2}}.$$

This implies

$$M(t) \geq (M(0)^{-\frac{p-1}{2}} - (p-1)(2\pi)^{-p} |\alpha|^{-\frac{p+1}{2}} \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda t)^{-\frac{2}{p-1}}$$

and therefore

$$\begin{aligned}
T_0 &\leq \inf \left\{ \frac{(2\pi)^p |\alpha|^{\frac{p+1}{2}}}{(p-1) \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda} M(0)^{-\frac{p-1}{2}}; \alpha \in \mathbb{C}, \operatorname{Re} \alpha \cdot \operatorname{Re} \lambda > 0 \right\} \\
&\leq \frac{(2\pi)^p}{(p-1) |\operatorname{Re} \lambda|} \left| \int_0^{2\pi} u_0(x) \int_0^x \overline{u_0(y)} dy dx \right|^{-\frac{p-1}{2}}.
\end{aligned}$$

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